FP3 June 15 Kprime 2

M-A

1. Solve the equation

$$2\cosh^2 x - 3\sinh x = 1$$

giving your answers in terms of natural logarithms. **(6)**

$c^2 - s^2 = 1 \Rightarrow c^2 = 1 + s^2$

$\therefore 2c^2 - 3s = 1$

$\Rightarrow 2(1+s^2) - 3s = 1$

$\therefore 2 + 2s^2 - 3s = 1$

$\therefore 2s^2 - 3s + 1 = 0$

$\therefore (s-1)(2s-1) = 0$

$\Rightarrow \sinh x = 1 \Rightarrow x = \text{arsinh}\, 1 = \ln(1+\sqrt{2})$

$\sinh x = \frac{1}{2} \Rightarrow x = \text{arsinh}\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)$

2. A curve has equation

$$y = \cosh x, \qquad 1 \leqslant x \leqslant \ln 5$$

Find the exact length of this curve. Give your answer in terms of e.

(5)

$y = \cosh x \Rightarrow \frac{\partial y}{\partial x} = \sinh x$

$$\text{length} = \int_1^{\ln 5} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} \, \partial x$$

$c^2 - s^2 = 1$

$$= \int_1^{\ln 5} \sqrt{1 + \sinh^2 x} \, \partial x$$

$$= \int_1^{\ln 5} \cosh x \, \partial x$$

$$= \Big[\sinh x\Big]_1^{\ln 5}$$

$$= \sinh(\ln 5) - \sinh(1)$$

$$= \frac{e^{\ln 5} - e^{-\ln 5}}{2} - \frac{e^1 - e^{-1}}{2}$$

$$= \frac{5 - \frac{1}{5}}{2} - \frac{e - \frac{1}{e}}{2}$$

$$= \frac{5 - \frac{1}{5} - e + \frac{1}{e}}{2} \times \frac{5e}{5e} = \frac{25e - \cancel{e} - 5e^2 + 5}{10e}$$

$$= \frac{24e - 5e^2 + 5}{10e}$$

3.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

(a) Find the eigenvalues of **A**. **(5)**

(b) Find a normalised eigenvector for each of the eigenvalues of **A**. **(5)**

(c) Write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^T\mathbf{A}\mathbf{P} = \mathbf{D}$. **(2)**

(a) $A - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (2-\lambda)\left((2-\lambda)^2 - 1\right) - (2-\lambda) = 0$

$\Rightarrow (2-\lambda)^3 - (2-\lambda) - (2-\lambda) = 0$

$\therefore (2-\lambda)^3 - 2(2-\lambda) = 0$

$\therefore (2-\lambda)\left((2-\lambda)^2 - 2\right) = 0$

$\therefore (2-\lambda)(\lambda^2 - 4\lambda + 2) = 0$

$\therefore \cancel{(2-\lambda)(} \Rightarrow 2-\lambda = 0 \therefore \lambda = 2$

$(2-\lambda)^2 - 2 = 0 \Rightarrow 2-\lambda = \pm\sqrt{2}$

$\therefore \lambda = 2 \pm \sqrt{2}$

$\therefore$ E. Vales are $\lambda = 2$

$\lambda = 2 \pm \sqrt{2}$

(b) Consider $\lambda = 2$:

$A\underline{x} = 2\underline{x}$

$$\therefore \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\therefore \begin{pmatrix} 2x+y \\ x+2y+z \\ y+2z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \begin{array}{l} \Rightarrow y=0 \\ \rightarrow x+z=0 \\ \Rightarrow y=0 \end{array}$$

$x = -z \,, \quad y = 0$

Let $x=1$, $z=-1$, $y=0$

$\therefore$ E.Vector $\underline{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Normalised $\hat{\underline{x}} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Consider $\lambda = 2+\sqrt{2}$:

$$\begin{pmatrix} 2x+y \\ x+2y+z \\ y+2z \end{pmatrix} = \begin{pmatrix} 2x+\sqrt{2}x \\ 2y+\sqrt{2}\,y \\ 2z+\sqrt{2}\,z \end{pmatrix}$$

$\Rightarrow$ $y = x\sqrt{2}$

$x + z = y\sqrt{2}$

$y = z\sqrt{2}$

~~$y = 1 \Rightarrow x = \frac{1}{\sqrt{2}} \Rightarrow z = \frac{1}{\sqrt{2}}$~~

$y = \sqrt{2} \Rightarrow x = 1 \Rightarrow z = 1$

$\therefore\ x = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

$\therefore\ y = x\sqrt{2} = z\sqrt{2}$

$\therefore$ normalise:

$x = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

$x = 2 - \sqrt{2}$

$\Rightarrow$ $\begin{pmatrix} 2x + y \\ x + 2y + z \\ y + 2z \end{pmatrix} = \begin{pmatrix} 2x - x\sqrt{2} \\ 2y - y\sqrt{2} \\ 2z - z\sqrt{2} \end{pmatrix}$

$\therefore\ y = -x\sqrt{2}$

$x + z = -y\sqrt{2}$

$y = -z\sqrt{2}$

$\therefore\ y = -z\sqrt{2} = -x\sqrt{2}$

$y = \sqrt{2} \Rightarrow z = -1 \Rightarrow x = -1$

$\therefore\ x = \begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix} \Rightarrow$ normal $x = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{2} \\ -1 \end{pmatrix}$

Question 3 continued

$$\therefore \lambda = 2 \Rightarrow \underset{\sim}{x} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\lambda\ 2-\sqrt{2} \Rightarrow \underset{\sim}{x} = \frac{1}{2}\begin{pmatrix}-1\\\sqrt{2}\\-1\end{pmatrix}$$

$$\lambda = 2+\sqrt{2} \Rightarrow \underset{\sim}{x} = \frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix}$$

(c) $$P = \begin{pmatrix}-1/2 & 1/\sqrt{2} & 1/2\\ \sqrt{2}/2 & 0 & \sqrt{2}/2\\ -1/2 & -1/\sqrt{2} & 1/2\end{pmatrix}$$

$$\cancel{P^TAP} = \begin{pmatrix}-1/2 & \sqrt{2}/2 & -1/2\\ 1/\sqrt{2} & 0 & -1/\sqrt{2}\\ 1/2 & \sqrt{2}/2 & 1/2\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}$$

$$= \left(\cancel{\frac{-2+\sqrt{2}}{2}}\right)$$

$$D = \begin{pmatrix}2-\sqrt{2} & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2+\sqrt{2}\end{pmatrix}$$

Q3

(Total 12 marks)

4. The curve C has equation

$$y = \frac{1}{\sqrt{x^2 + 2x - 3}}, \qquad x > 1$$

(a) Find $\int y \, dx$ **(3)**

The region R is bounded by the curve C, the x-axis and the lines with equations $x = 2$ and $x = 3$. The region R is rotated through 2π radians about the x-axis.

(b) Find the volume of the solid generated. Give your answer in the form $p\pi \ln q$, where p and q are rational numbers to be found. **(4)**

(a). $$\int \frac{1}{\sqrt{(x+1)^2 - 4}} \, dx$$

$$= \operatorname{arcosh}\left(\frac{x+1}{2}\right) + C$$

0.288

(b) $$V = \pi \int_2^3 y^2 \, dx$$

$$\therefore V = \pi \int_2^3 \frac{1}{(x+1)^2 - 4} \, dx$$

$\frac{2}{\cancel{8}}$

$\frac{1}{5}$

$$= \pi \left[\frac{1}{4} \ln \left| \frac{\cancel{x-2}\; x-1}{x+3} \right| \right]_2^3$$

$$= \pi \left[\frac{1}{4} \ln\left(\frac{1}{3}\right) - \frac{1}{4} \ln 1/5 \right]$$

$$= \frac{1}{4} \pi \ln (5/3)$$

Leave blank

5. The points A, B and C have position vectors $\begin{pmatrix}1\\3\\2\end{pmatrix}$, $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\1\\0\end{pmatrix}$ respectively.

(a) Find a vector equation of the straight line AB.

(2)

(b) Find a cartesian form of the equation of the straight line AB.

(2)

The plane Π contains the points A, B and C.

(c) Find a vector equation of Π in the form $\mathbf{r.n} = p$.

(4)

(d) Find the perpendicular distance from the origin to Π.

(2)

5(a) $AB = \begin{pmatrix}-1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}-2\\-3\\-1\end{pmatrix}$

~~(b)~~ $\therefore r = \lambda\begin{pmatrix}-2\\-3\\-1\end{pmatrix} + \begin{pmatrix}1\\3\\2\end{pmatrix}$

(b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2\lambda \\ 3-3\lambda \\ 2-\lambda \end{pmatrix}$

$\frac{1-x}{2} = \lambda$

$\frac{3-y}{3} = \lambda$

$2-z = \lambda$

$\therefore \quad \frac{1-x}{2} = \frac{3-y}{3} = 2-z$

$\times 6 \Rightarrow 3-3x = 6-2y = 12-6z$

(c) $AB = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$

$AC = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$\therefore AB \times AC = \begin{matrix} -2 & -3 & -1 & -2 & -3 & -1 \\ 1 & -2 & -2 & 1 & -2 & -2 \end{matrix}$

$= \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix}$

$\therefore r \cdot \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix}$

$= 8 - 5 = 3$

$\therefore r \cdot \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix} = 3$ ~~$r \cdot \hat{n} = \hat{n}$~~

(d) $\hat{n} = \frac{\sqrt{10}}{30} \begin{pmatrix} 4 \\ -5 \\ 7 \end{pmatrix}$

$r \cdot \hat{n} = \frac{\sqrt{10}}{30} \times 3 = \frac{\sqrt{10}}{10}$

$\therefore d = \frac{\sqrt{10}}{10}$

P 4 4 8 3 3 A 0 1 3 2 8

Turn over

6. The hyperbola H is given by the equation $x^2 - y^2 = 1$

(a) Write down the equations of the two asymptotes of H. **(1)**

(b) Show that an equation of the tangent to H at the point P ($\cosh t$, $\sinh t$) is

$$y\sinh t = x\cosh t - 1$$

(3)

The tangent at P meets the asymptotes of H at the points Q and R.

(c) Show that P is the midpoint of QR. **(3)**

(d) Show that the area of the triangle OQR, where O is the origin, is independent of t. **(3)**

6(a). $x = \pm y \quad \therefore \quad x = y$

$x = -y$

(b) $\frac{\partial y}{\partial x} = \frac{\partial y/\partial t}{\partial x/\partial t} = \frac{\cosh t}{\sinh t} = \coth(t)$

$\therefore y - y_1 = m(x - x_1)$

$\therefore y - \sinh t = \frac{\cosh t}{\sinh t}x - \frac{\cosh^2 t}{\sinh t}$

$\therefore y\sinh t - \sinh^2 t = x\cosh t - \cosh^2 t$

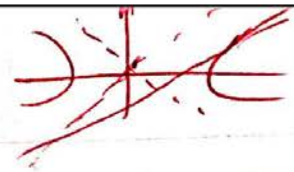
$\therefore y\sinh t = x\cosh t + \sinh^2 t - \cosh^2 t$

$\therefore y\sinh t = x\cosh t - (\cosh^2 t - \sinh^2 t)$

$\therefore y\sinh t = x\cosh t - 1$

as required.

Question 6 continued



~~(b)~~

(b) Consider $y = x$

$$\Rightarrow \quad x\sinh t = x\cosh t - 1$$

$$\therefore \quad (\sinh t - \cosh t)x = -1$$

$$\therefore \quad x = \frac{1}{\cosh t - \sinh t} = y$$

$$\therefore \text{ Let } Q\left(\frac{1}{\cosh t - \sinh t}, \frac{1}{\cosh t - \sinh t}\right)$$

$y = -x$

$$\Rightarrow \quad -x\sinh t = x\cosh t - 1$$

$$\therefore \quad x(\cosh t + \sinh t) = 1$$

$$\therefore \quad x = \frac{1}{\cosh t + \sinh t}$$

~~$y =$~~ $\therefore$ Let $R: \left(\frac{1}{\cosh t + \sinh t}, \frac{-1}{\cosh t + \sinh t}\right)$

Midpoint of QR:

$$\left(\frac{1}{2}\left(\frac{1}{\cosh t+\sinh t}+\frac{1}{\cosh t-\sinh t}\right),\ \frac{1}{2}\left(\frac{1}{\cosh t-\sinh t}-\frac{1}{c+s}\right)\right)$$

Let $\sinh t = s$ $\quad c^2-s^2=1$
$\cosh t = c$

Consider x coordinate of Midpoint QR:

$$x=\frac{1}{2}\left(\frac{1}{c+s}+\frac{1}{c-s}\right)=\frac{1}{2}\left(\frac{c-s}{(c+s)(c-s)}+\frac{(c+s)}{(c-s)(c+s)}\right)$$

$$=\frac{1}{2}\left(\frac{2c}{(c+s)(c-s)}\right)=\frac{1}{2}\left(\frac{2c}{c^2-s^2}\right)$$

$$=\frac{1}{2}\left(\frac{2c}{1}\right)=c=\cosh t$$

$\therefore$ x coordinate $=\cosh t = x_p$

Consider y coordinate of midpoint QR:

$$y=\frac{1}{2}\left(\frac{1}{c-s}-\frac{1}{c+s}\right)=\frac{1}{2}\left(\frac{c+s}{(c-s)(c+s)}-\frac{(c-s)}{(c+s)(c-s)}\right)$$

$$=\frac{1}{2}\left(\frac{c+s-c+s}{c^2-s^2}\right)=\frac{1}{2}\left(\frac{2s}{c^2-s^2}\right)=\frac{1}{2}\left(\frac{2s}{1}\right)$$

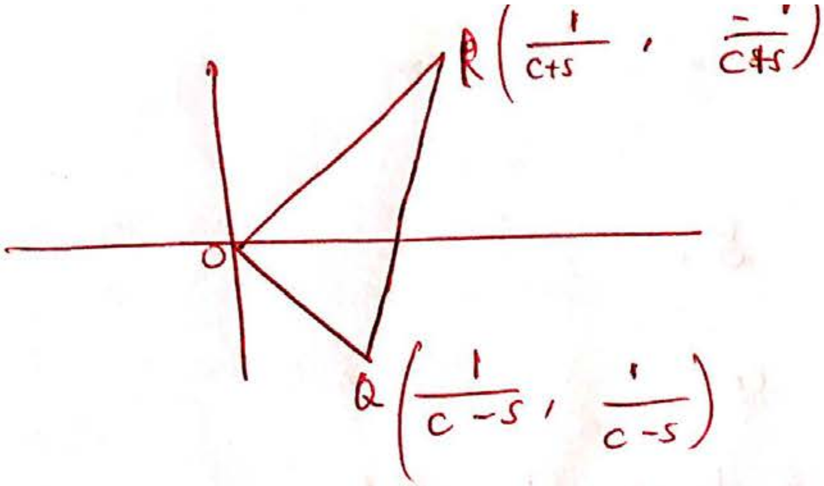
Question 6 continued

$= S = \sinh t = y_P$

$\therefore$ ~~$x_P = \cot$~~

$\therefore$ P is midpoint of QR

(d)



Let $c = \cosh t$
Let $s = \sinh t$

$$\text{Area} = \frac{1}{2} \times |OQ| \times |OR|$$

$$= \frac{1}{2} \cdot \sqrt{\left(\frac{1}{c-s}\right)^2 + \left(\frac{1}{(c-s)^2}\right)} \cdot \sqrt{\frac{1}{(c+s)^2} + \frac{1}{(c+s)^2}}$$

$$= \frac{1}{2} \cdot \sqrt{2} \cdot \sqrt{\frac{1}{(c-s)^2}} \cdot \sqrt{2} \sqrt{\frac{1}{(c+s)^2}}$$

$$= \frac{1}{2} \cdot \sqrt{2} \cdot \frac{1}{c-s} \cdot \sqrt{2} \cdot \frac{1}{c+s}$$

$$= \frac{1}{(c-s)(c+s)} = \frac{1}{c^2 - s^2} = 1$$

$\therefore$ Area is always one regardless of how 't' varies

$\therefore$ Area is independent of 't'

7. $$I_n = \int \sin^n x \, dx, \quad n \geqslant 0$$

(a) Prove that for $n \geqslant 2$

$$I_n = \frac{1}{n}(-\sin^{n-1}x\cos x + (n-1)I_{n-2})$$ **(4)**

Given that n is an odd number, $n \geqslant 3$

(b) show that

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \frac{(n-1)(n-3)\ldots 6.4.2}{n(n-2)(n-4)\ldots 7.5.3}$$ **(4)**

(c) Hence find $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^2 x \, dx$ **(3)**

7(a). $I_n = \int \sin^n x \, dx = \int \sin^{n-1} \sin x \, dx$

Let $u = \sin^{n-1} x \qquad u' = (n-1)(\sin^{n-2} x) \cos x$

$v' = \sin x \qquad v = -\cos x$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1) \int \cos^2 x \sin^{n-2} x \, dx$

since $\cos^2 x = 1 - \sin^2 x$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x - \sin^n x \, dx$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1) [I_{n-2} - I_n]$

$\therefore I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$\therefore (1+n-1) I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

$\therefore n I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

P 4 4 8 3 3 A 0 2 0 2 8

$$\therefore I_n = \frac{1}{n}\left(-\cos x \sin^{n-1}x + (n-1) I_{n-2}\right)$$

(b) Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$

$$I_n = \frac{1}{n}\left(\left[-\cos x \sin^{n-1} x\right]_0^{\pi/2} + (n-1) I_{n-2}\right)$$

$$\therefore I_n = \frac{(n-1)}{n} I_{n-2}, \quad I_{n-2} = \frac{n-3}{n-2} I_{n-4}, \ldots$$

$$\therefore I_n = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right) I_{n-4} = \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\left(\frac{n-5}{n-4}\right) I_{n-6} \ldots$$

$$\Rightarrow I_n = \frac{(n-1)(n-3)(n-5)\ldots 6\cdot 4\cdot 2}{n(n-2)(n-4)\ldots 7\cdot 5\cdot 3}$$

as required

Leave blank

Question 7 continued

(c) $\int_0^{\pi/2} \sin^5 x \cos^2 x \, dx$

~~let~~ $= \int_0^{\pi/2} \sin^5 x (1 - \sin^2 x) \, dx$

$= \int_0^{\pi/2} \sin^5 x - \sin^7 x \, dx$

$= \int_0^{\pi/2} \sin^5 x \, dx - \int_0^{\pi/2} \sin^7 x \, dx$

$= \frac{4 \cdot 2 \cdot 1}{5 \cdot 3 \cdot 1} - \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1}$

$= \frac{8}{15} - \frac{16}{35} = \frac{8}{105}$

8. The ellipse E has equation $x^2 + 4y^2 = 4$

(a) (i) Find the coordinates of the foci, F_1 and F_2, of E.

(ii) Write down the equations of the directrices of E.

(4)

(b) Given that the point P lies on the ellipse, show that

$$|PF_1| + |PF_2| = 4$$

(4)

A chord of an ellipse is a line segment joining two points on the ellipse.

The set of midpoints of the parallel chords of E with gradient m, where m is a constant, lie on a straight line l.

(c) Find an equation of l.

(6)

8(a). $\frac{x^2}{4} + y^2 = 1$

~~(i)~~ ~~e=~~ $1 = 4(1-e^2)$

$\therefore \frac{1}{4} = 1 - e^2 \Rightarrow e^2 = 3/4$

$\therefore e = \frac{\sqrt{3}}{2}$

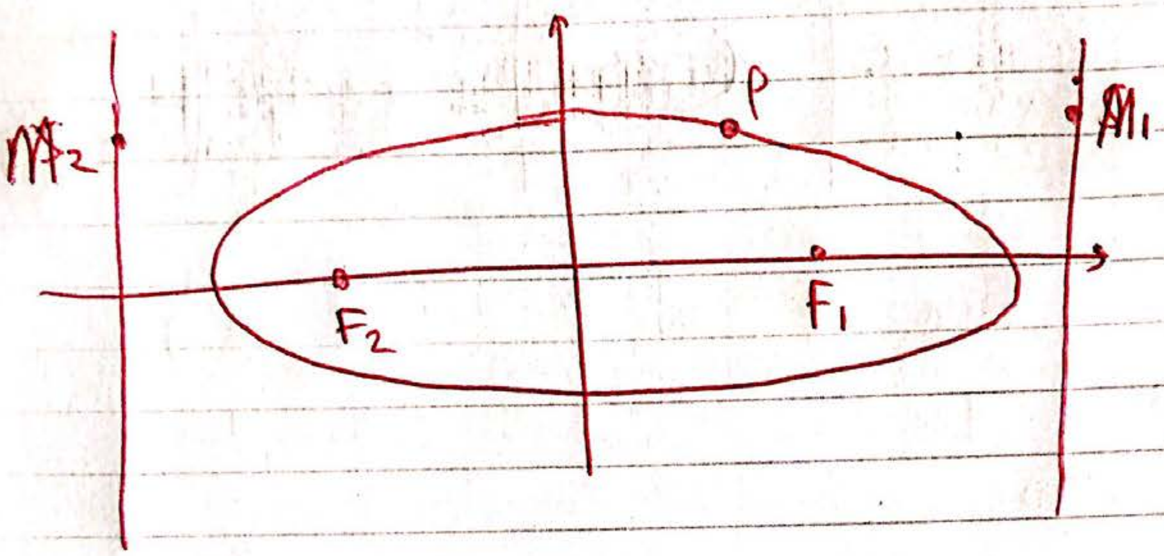
(i) ~~fo~~ $F_1(\sqrt{3}, 0)$

~~fo~~ $F_2(-\sqrt{3}, 0)$

(ii) $x = \pm \frac{2}{\frac{\sqrt{3}}{2}} = \pm \frac{4\sqrt{3}}{3}$

Question 8 continued

b)



$$e = \frac{PF_1}{PM_1}, \quad e = \frac{PF_2}{PM_2}$$

$$\therefore PF_1 = ePM_1 \quad \& \quad PF_2 = ePM_2$$

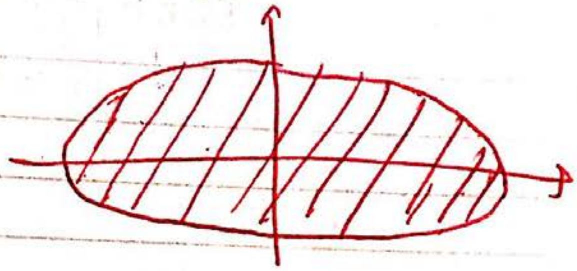
$$PM_1 + PM_2 = \frac{4\sqrt{3}}{3} + \frac{4\sqrt{3}}{3} = \frac{8\sqrt{3}}{3}$$

$$\therefore ePM_1 + ePM_2 = e \times \frac{8\sqrt{3}}{3}$$

$$\therefore PF_1 + PF_2 = e \times \frac{8\sqrt{3}}{3} = \frac{\sqrt{3}}{2} \times \frac{8\sqrt{3}}{3}$$

$$\therefore PF_1 + PF_2 = 4$$ as required

(C) $x^2 + 4y^2 = 4$



$\therefore \quad 2x + 8y \frac{\partial y}{\partial x} = 0$

$\therefore \quad \frac{\partial y}{\partial x} = \frac{-2x}{8y} = -\frac{x}{4y}$

$\Rightarrow \quad \frac{\partial y}{\partial x} = -\frac{x}{4y} = m$

$\therefore \quad -x = 4ym$

$\therefore \quad y = -\frac{x}{4m}$